A scale-invariant "discrete-time" Balitsky Kovchegov equation

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We consider a version of QCD dipole cascading corresponding to a finite number n of discrete ΔY steps of branching in rapidity. The disctretization scheme preserving the holomorphic factorizability and scale-invariance in position space of the dipole splitting function, we derive an exact recurrence formula from step to step which plays the rôle of a "discrete-time" Balitsky-Kovchegov equation. The BK solutions are recovered in the limit $n = \infty$ and $\Delta Y = 0$.

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I. INTRODUCTION

As was first suggested in [1], the distribution of partons inside a hadron, in the limit of high-energy, and fixed (large) Q^2 and large N_c, can be approximated by a cascade of colourless dipoles. Starting from this observation, and using the evolution equation describing the development of the cascade in rapidity Y, it was possible to obtain -in the leading logarithmic approximation- an evolution equation for the scattering amplitude of the dipole cascade on an "uncorrelated" target satisfying the unitarity constraint [2]. The dipole cascading formalism gives a derivation of QCD evolution equations obtained from the perturbative expansion in the leading logs approximation in energy. Both the linear regime corresponding to the Balitsky Fadin Kuraev and Lipatov (BFKL) evolution equation [3], and its non-linear extension, the Balitsky-Kovchegov (BK) equation [2, 4], find a convenient description in terms of dipoles. A general characteristics of the solutions of the BK equation was recently discussed in [5]. It was shown that an approximate geometrical scaling is the generic asymptotic energy property of the system, related to mathematical solutions in terms of traveling waves, and a general method of finding these solutions was developped.

The dipole cascading formulation of Refs.[1, 2] corresponds to a classical branching process where dipoles split with a probability distribution given by the BFKL kernel [3] expressed in transverse position space taking the role of (2-dimensional) space, the rapidity variable having the role of *time*. As such it belongs to the large family of random branching processes which is largely studied in statistical mechanics and in mathematics (see [5, 6, 7, 8, 9]). While the original QCD dipole formulation of [1, 2] is based on random branching, we remark that well-known classes of branching processes are considered with discrete steps in time [6], with interesting physical and mathematical properties, and applications. We want to address here the question of a similar discretization of dipole cascading.

The attractive feature of the QCD dipole approach is that -being formulated in the framework of the QCD perturbation theory- it is a well-defined stochastic fragmentation system. In particular the cascade vertices are uniquely given by the theory. Any departure from this scheme runs into a problem of serious ambiguities even if one imposes the condition that in the some limit one should recover the known perturbative results.

In the present paper we study one class of dicrete-in-rapidity dipole cascades whose vertices are scale-invariant in transverse space. In particular we retain the important feature of *holomorphic separability* which is present in the leading-logarithmic approximation. We show that such a theory naturally leads to discretization of the dipole cascade and that in the limit of a large number of rapidity steps one recovers the results of the leading-logarithmic approximation. One may hope that this exercise will help to understand the structure of gluon cascading beyond the lowest order of the perturbation theory.

The plan of our study is the following. In the next section 2, we derive a discretized version of the dipole cascading preserving scale-invariance and the holomorphic separability of the original BFKL kernel. In section 3 we derive the corresponding non-linear master equation, which plays de role of a "discrete time" Balitsky-Kovchegov (discrete-time BK) equation. In the next section 4, we use the method of traveling wave solutions in order to explore the solutions of the discrete-time BK equation when the number of steps goes to infinity. In particular we prove the convergence to the BK solutions when the rapidity step size goes in the same proportion to zero. In section 5, we draw some possible

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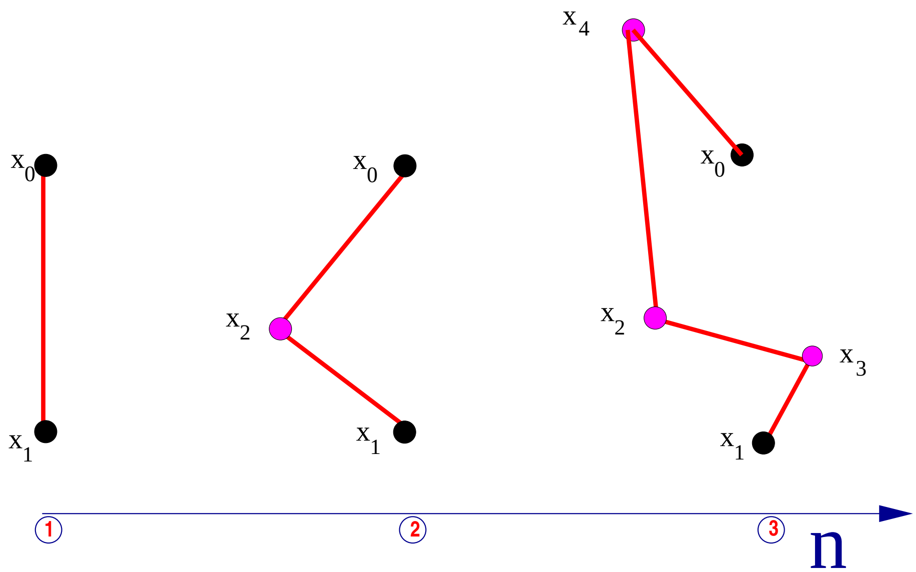

FIG. 1: *Discretized branching process of QCD dipoles.* The figure shows the rapidity steps for Y evolution of one dipole $x_0, x_1$for successive splittings at transverse space points x_2 and then x_3, x_4, etc...

interesting outcome of our discretisation procedure.

II. THE MASTER EQUATION

Starting from the the Balitsky-Kovchegov equation we show how the modification of the vertex leads to the discretization of the dipole cascade.

The BK equation reads:

$$\frac{d\mathcal{S}}{dY}(x_{01},Y) = \int d^2x_2\ \mathcal{K}(x_0,x_1;x_2)\ \left\{\mathcal{S}(x_{02},Y)\mathcal{S}(x_{12},Y) - \mathcal{S}(x_{01},Y)\right\}\ , \tag{1}$$

where $S(x_{01},Y)$ is the S-matrix element for the dipole-target amplitude, $x_{01} \equiv x_0 - x_1$ being the dipole transverse size. The BFKL kernel

$$\mathcal{K}(x_0,x_1;x_2)\ d^2x_2 = \frac{\alpha_s N_c}{2\pi^2}\ \frac{x_{01}^2}{x_{02}^2 x_{12}^2}\ d^2x_2 \tag{2}$$

has the classical interpretation [1] of a splitting probability in transverse space by unit of rapidity. Note that this kernel verifies simultaneously scale invariance, the $x_{02} \leftrightarrow x_{12}$ symmetry and holomorphic separability. To see this last property, it is worth using the 2-dimensional components of scaled variables $\xi^{(i)} \equiv x_i/x_{01}$ and transforming into complex variables, namely

$$\xi^{(1)} + i\xi^{(2)} = z,\ \ \xi^{(1)} - i\xi^{(2)} = \bar{z}\ . \tag{3}$$

Holomorphic separability means the factorization property of the kernel (2) into analytic (function of z) and antianalytic (function of $\bar{z}$) parts. It plays an important role in the conformal symetry properties of the BFKL amplitudes [10].

Let us consider a modified kernel preserving this property:

$$\tilde{\mathcal{K}}(x_0,x_1;x_2)\ d^2x_2 = \frac{\alpha_s N_c}{2\pi^2}\ \left(\frac{x_{01}^4}{x_{02}^2 x_{12}^2}\right)^{1-a}\ \frac{d^2x_2}{x_{01}^2} \equiv \frac{\alpha_s N_c}{2\pi^2}\ \left(z\bar{z}(1-z)(1-\bar{z})\right)^{a-1}\ \frac{dz d\bar{z}}{2i}, \tag{4}$$

and call

$$\mathcal{N} = \int d^2x_2\ \tilde{\mathcal{K}}(x_0,x_1;x_2)\ , \tag{5}$$

where $\mathcal{N}$ is finite for $0 < a < 1/2$.

We can now safely work separately on the two terms of (1) and write

$$\frac{1}{\mathcal{N}}\frac{d\mathcal{S}}{dY} = \left\{\int d^2x_2\ \mathcal{K}_d(x_0,x_1;x_2)\ \mathcal{S}(x_{02},Y)\mathcal{S}(x_{12},Y)\right\} - \mathcal{S}(x_{01},Y)\ , \tag{6}$$

where $\mathcal{K}_d \equiv \tilde{\mathcal{K}}/\mathcal{N}$ is now a properly normalized probability distribution.

Using the known [11] mathematical identity[1]

$$\frac{1}{2\pi i}\int dz d\bar{z}\ z^{(A-1)}\bar{z}^{(\tilde{A}-1)}\,(1-z)^{(B-1)}\,(1-\bar{z})^{(\tilde{B}-1)} = \frac{\Gamma\,(A)\ \Gamma\,(B)\ \Gamma(1-\tilde{A}-\tilde{B})}{\Gamma(1-\tilde{A})\ \Gamma(1-\tilde{B})\ \Gamma\,(A+B)}\ , \tag{7}$$

we obtain the expression for the normalization

$$\mathcal{N} = \frac{\alpha_s N_c}{2\pi}\times\frac{\Gamma^2\,(a)\ \Gamma(1-2a)}{\Gamma^2(1-a)\ \Gamma\,(2a)}\ , \tag{8}$$

and the following expression for the probability distribution

$$\mathcal{K}_d(x_0,x_1;x_2)\ d^2x_2 = \frac{1}{\pi}\left(\frac{x_{01}^4}{x_{02}^2x_{12}^2}\right)^{1-a}\ \frac{d^2x_2}{x_{01}^2}\times\left\{\frac{\Gamma^2\,(a)\ \Gamma(1-2a)}{\Gamma^2(1-a)\ \Gamma\,(2a)}\right\}^{-1}\ . \tag{9}$$

Choosing a finite interval

$$\mathcal{N}\Delta Y \equiv \Delta n = 1 \tag{10}$$

and considering $n\in\mathbb{N}, n\gg 1$ allows one to transform Eq. (6) into a finite difference equation

$$\mathcal{S}_{n+1}(x_{01}) - \mathcal{S}_n(x_{01}) = \int d^2x_2\ \mathcal{K}_d(x_0,x_1;x_2)\left\{\mathcal{S}_n(x_{02})\mathcal{S}_n(x_{12}) - \mathcal{S}_n(x_{01})\right\}\ , \tag{11}$$

leading to (the distribution $\mathcal{K}_d$, see (9), being normalized to 1)

$$\mathcal{S}_{n+1}(x_{01}) = \int \mathcal{K}_d(x_0,x_1;x_2)\ d^2x_2\times\mathcal{S}_n(x_{02})\ \mathcal{S}_n(x_{12})\ . \tag{12}$$

As clear enough from its formal structure, Eq.(12) is a "discrete time" version of the Balitsky-Kovchegov equation (1). It remains to be proven that it leads back to the BK equation when going to its continuous limit, which will be defined in the next section.

Indeed, Eq.(12) has the typical structure for S-matrix elements defined for a branching process (or tree structure) with discrete steps of time evolution. see Fig.1. At each step $n\to n+1$ a dipole of size x_{01} splits into two dipoles of sizes x_{02}, x_{12} at the point x_2 with a 2-dimensional probability distribution $\mathcal{K}_d(x_0,x_1;x_2)$. In this description, the formula (10) determines the length of the "rapidity veto" $\Delta Y = 1/\mathcal{N}$. The dependence of ΔY on a is shown in Figure 2.

III. THE RECURRENCE STRUCTURE IN MOMENTUM SPACE

An even simpler form of equation (12) can be obtained in momentum space, when one considers solutions independent of the impact-parmeter. It reveals even better the recurrence structure of the discrete-time BK equation. Using a 2-dimensional Fourier transform, one defines

$$\tilde{\mathcal{S}}_n^{(a)}(\mathbf{k}) \equiv \frac{\int\frac{d^2x_{01}}{x_{01}^2}\ (x_{01}^2)^a\ e^{i\mathbf{k}\cdot\mathbf{x_{01}}}\ S_n(x_{01})}{\int\frac{d^2x_{01}}{x_{01}^2}\ (x_{01}^2)^a\ e^{i\mathbf{k}\cdot\mathbf{x_{01}}}} = \frac{\int dx\ J_0(kx)\ S_n(x)\ x^{2a-1}}{\int dx\ J_0(kx)\ x^{2a-1}}, \tag{13}$$

[1] There is one condition for (7) to be valid, namely $A-\tilde{A}, B-\tilde{B}\in\mathbb{Z}$, which also ensures that formula (7) is symetric in the interchange $A\to\tilde{A}, B\to\tilde{B}$.

with

$$\int dx\ J_0(kx)\ x^{2a-1} = 2^{2a-1}\ k^{-2a} \frac{\Gamma(a)}{\Gamma(1-a)}\ . \tag{14}$$

working in the context of an impact parameter independent framework. Note that $\tilde{\mathcal{S}}_n^{(a)}(\mathbf{k})$, can be obtained from a convolution of the S-matrix element in momentum space by the Fourier transform of the the weight $(x_{01}^2)^a$ in the integrand of (13).

It is straightforward to infer from (12) and (13) the following relation

$$\tilde{\mathcal{S}}_{n+1}^{(2a)}(\mathbf{k}) = \left(\frac{\Gamma(1-a)}{\Gamma\,(a)}\right)^2 \int \frac{d^2x_{02}}{x_{12}^2}\ \frac{d^2x_{12}}{x_{12}^2} (x_{02}^2)^a\ (x_{12}^2)^a\ e^{i\mathbf{k}\cdot(\mathbf{x_{02}}+\mathbf{x_{21}})}\ \mathcal{S}_n(x_{02})\ \mathcal{S}_n(x_{12})\ , \tag{15}$$

where the substitution of the integration variable $x_2 \to x_{12}$, with Jacobian unity, has been performed.

Then, the discrete-time BK equation (12) can be rewritten in a particular simple way as

$$\tilde{\mathcal{S}}_{n+1}^{(2a)}(\mathbf{k}) = \left\{\tilde{\mathcal{S}}_n^{(a)}(\mathbf{k})\right\}^2\ , \tag{16}$$

which clearly expresses the nature of the non-linear recurrence relation from step to step. this recurrence structure is also clear when expressed in terms of transition matrix elements with the same conventional notation as (13)

$$\tilde{\mathcal{T}}_{n+1}^{(2a)}(\mathbf{k}) = 2\ \tilde{\mathcal{T}}_n^{(a)}(\mathbf{k}) - \left\{\tilde{\mathcal{T}}_n^{(a)}(\mathbf{k})\right\}^2\ . \tag{17}$$

IV. HIGH ENERGY LIMITS

A. Linear regime

Let us examine the properties of the master equation (12) (also (16)). Considering first the linearized form of Eq. (12) near $\mathcal{S} \sim 1$ we have

$$\mathcal{T}_{n+1}(x_{01}^2) = \int d^2x_2\ \mathcal{K}_d(x_0, x_1; x_2)\ \left\{\mathcal{T}_n(x_{02}^2) + \mathcal{T}_n(x_{12}^2)\right\}\ , \tag{18}$$

where $\mathcal{T} \equiv 1 - \mathcal{S}$ is the transition matrix element. Using the complex scaled variables (3) one writes

$$\mathcal{T}_{n+1}(t) = 2\int dz d\bar{z}\ \mathcal{K}_d(z,\bar{z})\ \mathcal{T}_n(tz\bar{z})\ , \tag{19}$$

where

$$\mathcal{K}_d(z,\bar{z}) = \left(\frac{\Gamma^2(1-a)\ \Gamma\,(2a)}{\pi\Gamma^2\,(a)\ \ \Gamma(1-2a)}\right)\ \left\{z\bar{z}(1-z)(1-\bar{z})\right\}^{a-1}\ . \tag{20}$$

and the factor 2 comes from the $z \rightleftarrows 1-z$ symmetry of $\mathcal{K}_d$.

We now introduce the Mellin transform representation

$$\mathcal{T}_n(t) = \int_{\mathcal{C}} \frac{d\gamma}{2i\pi}\ t^\gamma\ \tilde{\mathcal{T}}_n(\gamma)\ , \tag{21}$$

and easily derive the recursive relation

$$\tilde{\mathcal{T}}_{n+1}(\gamma) = \tilde{\mathcal{T}}_n(\gamma)\ e^{\ \chi^{(a)}(\gamma)}, \tag{22}$$

where, using (7),

$$\chi^{(a)}(\gamma) \equiv \log\left(2\ \frac{\Gamma\,(a+\gamma)\ \Gamma\,(1-a)\ \Gamma(1-2a-\gamma)\ \Gamma(2a)}{\Gamma(a)\ \Gamma(1-a-\gamma)\ \Gamma\,(1-2a)\ \ \Gamma\,(2a+\gamma)}\right)\ . \tag{23}$$

Hence, the solution of the linearized equation is simply

$$\tilde{\mathcal{T}}_n(\gamma) = \tilde{\mathcal{T}}_0(\gamma)\ \exp\left\{n\chi^{(a)}(\gamma)\right\} = \tilde{\mathcal{T}}_0(\gamma)\ \exp\left\{\mathcal{N}Y\ \chi^{(a)}(\gamma)\right\}\ , \tag{24}$$

where Y is the total rapidity interval and $\mathcal{N}$ is given in (8).

Let us now consider the limit $a \to 0$. A straightforward algebra gives

$$\mathcal{N}\ \chi^{(a)} = \left\{\frac{1}{a}\ \frac{\alpha_s N_c}{\pi} + \mathcal{O}(a)\right\} \times \left\{a\left[2\Psi(1) - \Psi(\gamma) - \Psi(1-\gamma)\right] + \mathcal{O}(a^2)\right\} = \left[\frac{\alpha_s N_c}{\pi}(2\Psi(1) - \Psi(\gamma) - \Psi(1-\gamma))\right] + \mathcal{O}(a)\ , \tag{25}$$

where one recognizes in the brackets the BFKL kernel function $\chi(\gamma)$. Hence, from (24)

$$\tilde{\mathcal{T}}_n(\gamma) = \tilde{\mathcal{T}}_0(\gamma) \exp\left\{na\left[(2\Psi(1) - \Psi(\gamma) - \Psi(1-\gamma)\right]\right\} = \tilde{\mathcal{T}}_0(\gamma)\ \exp\left\{\frac{\alpha_s N_c}{\pi}\chi(\gamma)\ Y\right\}, \tag{26}$$

in which we recover the solution of the BFKL equation.

B. Non-linear regime

Let us now consider, in the spirit of [5] for the Balitskii-Kovchegov equation, the asymptotic solutions of the master equations (12) or (16) in the continuous limit, i.e. for a large number n of steps of the cascade. Asymptotic solutions we are discussing now may be valid only at really extreme energies. Nevertheless we feel that they may be of some interest. Indeed, as we shall see now by inspecting the general properties of the full non-linear equations for arbitrary a, the master equations (12) (or equivalently (16)) give the same solutions at large n than the Balitsky-Kovchegov equation in the limit $a \to 0$ or, in more mathematical terms, stay in the same universality class. We will also extend our study to another type of continuous limit, namely when n is large but a is kept fixed.

Let us start from the expressions (21) and (24) giving the solution of the linearised problem. One has

$$\mathcal{T}_n(t) = \int_{\mathcal{C}} \frac{d\gamma}{2i\pi}\ t^{\gamma}\ \tilde{\mathcal{T}}_0(\gamma)\ \exp\left\{\mathcal{N}Y\ \chi^{(a)}(\gamma)\right\}\ , \tag{27}$$

where $\chi^{(a)}$ has been obtained in (23). We will now follow mathematical arguments used for the solution of discrete non-linear equations appearing in a statistical mechanic context [6]. They are similar than those used in [5] for the asymptotic solutions of the Balitsky-Kovechegov equations and are part of more general mathematical results on non-linear equations, such as for the Fisher and Kolmogorov, Petrovsky, Piscounov equations [7, 8] of "pulled front" type [9].

Similarly to the arguments developped in [5], the solution of the non-linear equation equations (12) or (16) are traveling waves whose expressions can be obtained starting from the linear equation (27).

Let us reinterpret (27) as a linear superposition of waves:

$$\mathcal{T}_n(t) = \int_{\mathcal{C}} \frac{d\gamma}{2i\pi}\ \tilde{\mathcal{T}}_0(\gamma)\ \exp\left\{-\gamma(L_{\rm WF} + vn) + n\ \chi^{(a)}(\gamma)\right\}\ , \tag{28}$$

where $L \equiv \log 1/t$ and $L_{\rm WF} \equiv L - vn$ is looked for as the scaling variable determining the region moving with the traveling wave front. $\chi^{(a)}(\gamma)$ defines the dispersion relation of the linearized equation. In particular, each partial wave of wave number γ has a *phase velocity*

$$v_{\varphi}(\gamma) = \frac{\chi^{(a)}(\gamma)}{\gamma} \tag{29}$$

whose expression is found by imposing that the exponential factor in Eq.(28) be independent of n for $v = v_{\varphi}(\gamma)$. By contrast, the *group velocity* is defined by the saddle point γ^* of the exponential phase factor

$$v^* = \left.\frac{d\chi^{(a)}}{d\gamma}\right|_{\gamma^*} \equiv v_g\ . \tag{30}$$

The key point of the mathematical derivation of the asymptotic solution of the non-linear equation is that, for appropriate initial conditions [5, 6], the critical regime at $\gamma = \gamma^*$ is selected by the non-linear damping.

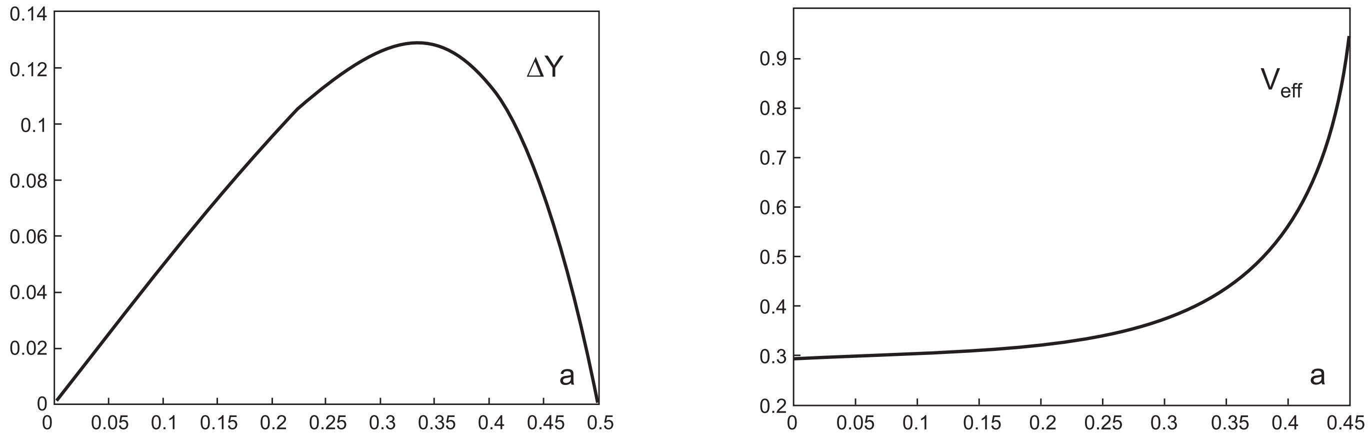

FIG. 2: *Features of the "discrete time" solutions.* Left: Effective rapidity step ΔY in units of $\frac{\alpha_s N_c}{2\pi}$ versus a. Right: Effective slope $v_Y = v^* \mathcal{N}$ (see formula (34)) at $\alpha_s = .1$ versus a.

The velocity v of the front is defined by

$$v^* = \frac{\chi^{(a)}(\gamma^*)}{\gamma^*} = \min_\gamma \frac{\chi^{(a)}(\gamma)}{\gamma} \ , \tag{31}$$

where the value of γ^* is determined from the equation

$$\frac{d\chi^{(a)}}{d\gamma} = \frac{\chi^{(a)}(\gamma)}{\gamma}\bigg|_{\gamma^*} \ . \tag{32}$$

Indeed, in this case, the group velocity is identical to the minimum of the phase velocity[2]. Now, since

$$n = Y/\Delta Y = Y\mathcal{N} \tag{33}$$

we can write $v^* n = v_Y Y$, with

$$v_Y = v^* \ \mathcal{N}. \tag{34}$$

Reporting these results in Eq.(28) gives the dominant term in the asymptotic expression for the form of the front:

$$\mathcal{T}(Y,t) \sim \tilde{\mathcal{T}}_0(\gamma^*) \ \exp\left\{-\gamma^* \log 1/t + \mathcal{N}\chi^{(a)}(\gamma^*)Y\right\} \ . \tag{35}$$

Note that two more "universal prefactors" (ie. independent of the initial conditions and of the precise form of the non-linear damping terms) can be obtained [5, 9] in the asymptotic expansion of the solution.

From (35), it is clear that the same result than for the Balitsky-Kovchegov equation will be obtained in the limit $a \to 0$, due to the equivalence of the linear kernels and the universality properties due to the the non-linearities. This achieves the proof.

Let us consider our high energy limits for $n \to \infty$ with $a \neq 0, 1/2$ kept fixed. Some features of the asymptotic solutions are displayed in Fig.2. As shown in the left part of the figure, the rapidity step ΔY is displayed as a function of a for a given value of $\alpha_s = .1$. There is a maximum of the reachable rapidity step for $a \sim .3$ whose value depends on α_s. At this stage this remains an intringuing feature of our scheme of discretization. The right hand part of the figure gives the effective slope (or "intercept") in rapidity, which, starting from the BK value stays constant till approximately $a = .25$ and then grows. Note that the energy momentum constraint on the number of steps at given rapidity range cannot be taken into account in this continuous limit, as will be discussed in the next section.

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[2] The relation $v_g = v_\varphi$, in analogy with wave physics, has been written in Ref.[12].

V. CONCLUSION AND OUTLOOK

To summarize the main results of our paper, we have considered a "discrete time" version of QCD dipole brancing process characterizing the approximation of the perturbative expansion in the leading logs of the energy. This is obtained by a modification of the splitting probability for finite steps in rapidity preserving the holomorphic factorizability of the initial BFKL kernel.

Our main result is the derivation of an exact counterpart of the BK saturation equation in terms of recurrence formulae of quite simple form, expressed both in position space (12) or in a (modified) fourier-transformed space (16).

We checked that we recover the BFKL and BK equations in the appropriate $\Delta Y \to 0$ continuous limit while the high energy asymptotics for $\Delta Y \neq 0$ have been investigated.

The principal outlook of our study concerns the problem of subasymptotics. Indeed, the master equations (12) and (16) allow for an iterative solution of the non-linear evolution problem. Inserting any initial condition $\mathcal{T}_0$ in (12) or in (modified) Fourier transform for (16), it is possible to generate the full solution, at each step of evolution. In particular, as explained in Section 2, for $a \neq 0, 1/2$ the steps of the cascade are separated by a finite distance ΔY in rapidity. Therefore, for a given total energy, the number of steps in the cascade are limited by $n_{max} \approx Y/\Delta Y$, where Y is the total available rapidity. Hence these equations give a convenient way to examine the effect of a limited number of steps, and thus of dipoles, in the cascading process. Indeed, the gluon cascade description of [1, 2] can only be justified in the leading logarithmic approximation. The investigations of higher order corrections shows [13] that they tend to limit the number of emitted gluons[3]. One possibility to take this effect into account is to forbid the emission of gluons which are too close in rapidity [13, 14]. Since the emitted gluons are at the source of dipole splitting [1], it is expected that the formation of dipoles is similarly limited in rapidity. Such a "rapidity veto" can -in turn- be approximated by a discretized cascade where gluon (or dipole) emissions is separated by a finite distance in rapidity. We thus feel that our equations (12,16)) could give an efficient way to investigate this problem. It also can be useful to develop the parallel with statistical mechanics properties which appeared to be recently fruitful [5].

When completing the writing of the present work, the paper [15] has appeared, claiming that the solution of an evolution with discrete rapidity intervals may lead to a chaotic behaviour. It stems from a toy model of the BK equation in the form of a logistic map. We note that our equation (17) has the structure of a logistic map if one can neglect the shift $(a) \to (2a)$ in the weight factor. However the "Malthusian parameter" would be then 2 instead of 3.77 (see the discussion in Ref.[15]) and thus is smaller than the value leading to chaos. Since our equation (17) is a consistent discretization of the BK equation, the investigation of its convergence (or not) to the mean fied problem deserves to be studied.

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